Chapter 9

Central Potentials

In this Chapter we will study the Schroedinger equation for a particle in 3 dimensions and subjected to a central potential $V(r)$, thus depending only on the distance from the center of coordinates $r = \sqrt{x^2 + y^2 + z^2}$. This is a case of paramount importance in physics, since potentials such as the Coulomb interaction intrinsically have radial symmetry as in this case.

9.1 The Hamiltonian

Analogously to the classical case, the Hamiltonian takes the form:

$$\hat{H} = \frac{\hat{p}^2}{2M} + V(|\vec{r}|), \tag{9.1.1}$$

with the first term corresponding to the kinetic energy,

$$\hat{K} = \frac{\hat{p}^2}{2M} = -\frac{\hbar^2}{2M}\nabla^2\,, \tag{9.1.2}$$

where we have called M the mass of the particle. Since the potential depends only on the distance $r = |\vec{r}|$, and not on the angle, it is clear that when rotating the coordinate system of an arbitrary angle the potential energy will not change. This implies that the potential energy commutes with the rotation operator $\hat{D}(\theta)$ and in turn that also the components of the orbital angular momentum must commute with $\hat{V}$:

$$[\hat{L}_\alpha, \hat{V}] = 0\,, \tag{9.1.3}$$

and as a consequence the orbital angular momentum squared $\hat{L}^2$ will also commute with the potential energy

$$[\hat{L}^2, V(\hat{x})] = 0\,. \tag{9.1.4}$$

Moreover, the kinetic term also commutes with the rotation operator $\hat{D}(\theta)$, since $\hat{p}^2$ is the norm of the vector $\hat{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$ and by construction we have seen that rotations in any direction do not change the norm of vectors, thus

$$[\hat{L}_\alpha, \hat{p}^2] = 0, \quad [\hat{L}^2, \hat{p}^2] = 0\,. \tag{9.1.5}$$

It then follows that the whole Hamiltonian commutes with the orbital angular momentum operators

$$[\hat{L}_\alpha, \hat{H}] = 0 \tag{9.1.6}$$

$$[\hat{L}^2, \hat{H}] = 0. \tag{9.1.7}$$

Because of the last two relations, we can immediately deduce that the states $|l, m\rangle$ we have found in the previous Chapter are also eigenstates of the Hamiltonian, and that a generic eigenstate of the Hamiltonian is therefore of the form we previously found:

$$\langle \vec{r}|l, m\rangle = \Phi^l(r) Y^l_m(\theta, \phi)\,. \tag{9.1.8}$$

In the previous Chapter, we had seen that the radial part, $\Phi^l(r)$, is an arbitrary function that cannot be determined diagonalizing the orbital angular momentum only. In this Chapter, we will show that the radial function satisfies a one-dimensional Schrödinger equation (for the radial variable r) with a modified potential energy, that we can call $V_{\text{eff}}(r)$ and that we will determine in the following.

Exercise 1.1 In this exercise, we will formally show that the Hamiltonian for a particle in a central potential commutes with the orbital angular momentum operators.
We will do this derivation in steps. We first start by examining the commutation relations with the kinetic energy operator.

1. First show the $\hat{L}_z$ has the following commutation relations with the components of the canonical momentum by using the classical definition.

$$\left[\hat{L}_z, \hat{p}_x\right] = i\hbar\hat{p}_y, \quad \left[\hat{L}_z, \hat{p}_y\right] = -i\hbar\hat{p}_x, \quad \left[\hat{L}_z, \hat{p}_z\right] = 0, \tag{9.1.9}$$

2. Generalize this result to show

$$\left[\hat{L}_i, \hat{p}_j\right] = i\hbar\varepsilon_{ijk}\hat{p}_k \tag{9.1.10}$$

3. Then using your previous result along with the identity $[A, BC] = B[A, C] + [A, B]C$ show that

$$\left[\hat{L}_i, \hat{p}^2\right] = 0, \quad \hat{p}^2 = \hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2 \tag{9.1.11}$$

   where $i \in \{x, y, z\}$.

4. Using the result of the previous question and the same identity, show that indeed the orbital momentum operators commute with the kinetic energy operator,i.e.

$$\left[\hat{L}_i, \frac{\hat{p}^2}{2m}\right] = 0 \quad \forall i \in \{x, y, z\}, \quad \left[\hat{L}^2, \frac{\hat{p}^2}{2m}\right] = 0 \tag{9.1.12}$$

We now follow a very similar for the potential operator.

1. Show that the z-component of the orbital angular momentum has the following commutation relations

$$\left[\hat{L}_z, \hat{x}\right] = i\hbar\hat{y}, \quad \left[\hat{L}_z, \hat{y}\right] = -i\hbar\hat{x}, \quad \left[\hat{L}_z, \hat{z}\right] = i\hbar\hat{z} \tag{9.1.13}$$

2. Generalize this result to
$$\left[\hat{L}_i, \hat{q}_j\right] = i\hbar\varepsilon_{ijk}\hat{q}_k, \tag{9.1.14}$$
where $\hat{q}_1 = \hat{x}, \hat{q}_2 = \hat{y}, \hat{q}_3 = \hat{z}$.

3. Show that
$$\left[\hat{L}_i, \hat{q}^2\right] = 0 \quad \forall i \in \{1,2,3\}, \quad \left[\hat{L}^2, \hat{q}^2\right] = 0 \tag{9.1.15}$$

4. Since we are working with central potentials, the potential energy is a function of r. However, let us further consider that it depends on r^2 such that $V(r^2)$. By expanding the potential in a power series of x^2, show that
$$\left[\hat{L}_i, V(\hat{r}^2)\right] = 0 \quad \forall i \in \{x,y,z\}, \quad \left[\hat{L}^2, (\hat{r}^2)\right] = 0 \tag{9.1.16}$$

Thus we have indeed shown that the Hamiltonian commutes with the angular orbital momentum operators.

9.2 The Kinetic Energy

The expression of the kinetic energy can be greatly simplified using the angular momentum operator. In order to do so, we consider the important identity

$$\hat{L}^2 = \hat{r}^2\hat{p}^2 - (\hat{r}\cdot\hat{p})^2 + i\hbar\hat{r}\cdot\hat{p}, \tag{9.2.1}$$

whose demonstration is straightforward but a bit lengthy. The above relation can be projected onto a ket in spherical coordinates space, $|r\rangle = |r,\theta,\phi\rangle$, and we obtain

$$\langle r|\,\hat{L}^2\,|\psi\rangle = \langle r|\left[\hat{r}^2\hat{p}^2 - (\hat{r}\cdot\hat{p})^2 + i\hbar\hat{r}\cdot\hat{p}\right]|\psi\rangle\ . \tag{9.2.2}$$

However, the last term is easily expressed in spherical coordinates, recalling that

$$\begin{aligned} r\frac{\partial}{\partial r} &= r\left(\frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} + \frac{\partial z}{\partial r}\frac{\partial}{\partial z}\right) \\ &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \\ &= \mathbf{r}\cdot\nabla, \end{aligned}$$

thus

$$\langle r|\,\hat{r}\cdot\hat{p}\,|\psi\rangle = -i\hbar\mathbf{r}\cdot\nabla\,\langle r|\psi\rangle = -i\hbar\left(r\frac{\partial}{\partial r}\right)\langle r|\psi\rangle\,. \tag{9.2.3}$$

Similarly, for the second term,

$$\langle r|\,(\hat{r}\cdot\hat{p})^2\,|\psi\rangle = \langle r|\,[\hat{r}\cdot\hat{p}]\,[\hat{r}\cdot\hat{p}]\,|\psi\rangle = -\hbar^2\left(r\frac{\partial}{\partial r}\right)^2\langle r|\psi\rangle\,. \tag{9.2.4}$$

Combining all these results, we arrive at the following expression for the kinetic energy operator in the Hamiltonian:

$$\frac{1}{2M}\langle r|\,\hat{p}^2\,|\psi\rangle = -\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\langle r|\,\psi\rangle + \frac{\langle r|\,\hat{L}^2\,|\psi\rangle}{2Mr^2}. \tag{9.2.5}$$

Physically speaking, this kinetic energy contains two parts that are easy to recognize: a term corresponding to the rotational kinetic energy (the second term) and another term that corresponds to the radial kinetic energy.

Exercise 1.2 We want to show that

$$\hat{L}^2 = \hat{r}^2\hat{p}^2 - (\hat{r}\cdot\hat{p})^2 + i\hbar\hat{r}\cdot\hat{p}\,. \tag{9.2.6}$$

Consider the classical definition of $\hat{\mathbf{L}} = \hat{\mathbf{r}}\times\hat{\mathbf{p}}$, expressed as follows

$$(\hat{\mathbf{r}}\times\hat{\mathbf{p}})_i = \sum_{jk}\epsilon_{ijk}\hat{r}_j\hat{p}_k\,. \tag{9.2.7}$$

We will also use the identity involving the product of two Levi-Civita tensors (which you may prove if you wish),

$$\sum_i \epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\,. \tag{9.2.8}$$

Using these two facts, show the above result.

9.3 The Radial Equation

We have previously seen that the Hamiltonian commutes with $\hat{L}^2$, thus the eigenfunctions of the Hamiltonian must also be eigenfunctions of $\hat{L}^2$ and all the components $\hat{L}_\alpha$ of the angular momentum. The time-independent Schrödinger equation

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)\psi(r) + \frac{\langle r|\,\hat{L}^2\,|\psi\rangle}{2Mr^2} + V(r)\psi(r) = E\psi(r), \tag{9.3.1}$$

can be then simplified noticing that $\hat{L}^2$ is acting on its eigenstate, thus

$$-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)\psi(r) + \frac{\hbar^2 l(l+1)}{2Mr^2}\psi(r) + V(r)\psi(r) = E\psi(r). \tag{9.3.2}$$

Moreover, we have seen that all eigenfunctions of the Hamiltonians are of the separable form

$$\psi(r) = \Phi^l(r)Y^l_m(\theta,\phi), \tag{9.3.3}$$

where $\Phi^l(r)$ is a radial function to be determined, and $Y^l_m(\theta,\phi)$ are the spherical harmonics we have derived previously.

Substituting Eq. (9.3.3) into Eq. (9.3.2), we have that the following differential equation has to be satisfied by the radial part:

$$\left[-\frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{l(l+1)\hbar^2}{2Mr^2} + V(r)\right]\Phi^l(r) = E\Phi^l(r). \tag{9.3.4}$$

Notice that from the beginning we have dropped the index m in the radial part ($\Phi^l_m(r) \to \Phi^l(r)$), and it is clear from the equation above that indeed Φ depends only on l. This differential equation can be further simplified making the substitution

$$\Phi^l(r) = \frac{u^l(r)}{r}, \tag{9.3.5}$$

yielding an effective one-dimensional Schrödinger equation:

$$\left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial r^2} + \left(\frac{l(l+1)\hbar^2}{2Mr^2} + V(r)\right)\right] u^l(r) = E u^l(r). \tag{9.3.6}$$

9.4 Properties of the radial wave function

The 1D Schrödinger equation for the radial wave function $u^l(r)$ is an important result of the previous section, because it allows us to map the complicated 3D Schrödinger equation on a case that is easier to analyze, both for numerical and analytical studies. The radial equation is entirely equivalent to a one-dimensional Schrödinger equation in the radial variable, for a particle moving in an effective potential

$$V^l_{\text{eff}}(r) = \frac{l(l+1)\hbar^2}{2Mr^2} + V(r). \tag{9.4.1}$$

The term $\frac{l(l+1)\hbar^2}{2Mr^2}$ is known as the centrifugal barrier and it is a repulsive force that increases for larger angular momenta l. Intuitively, what happens then is that at higher angular momenta the effective force felt by the atom increases, and, as we will see, states of large l correspond to wave functions in which the atomic density is "pushed away" from the origin. The radial wave function has also other general properties that we can deduce here.

9.4.1 Normalization

We start recalling that the overall wave function of the problem has been separated into a radial and an angular part, in a way that

$$\psi(r) = \Phi^l(r) Y^l_m(\theta, \phi), \tag{9.4.2}$$

$$= \frac{u^l(r)}{r} Y^l_m(\theta, \phi), \tag{9.4.3}$$

and that the overall normalization for the wave function is

$$\int dr |\psi(r)|^2 = \int_0^\infty dr r^2 |\Phi^l(r)|^2 \int d\Omega |Y^l_m(\theta, \phi)|^2, \tag{9.4.4}$$

and that since the spherical harmonics are already normalized to 1 over the angular variables (see discussion in the previous Chapter), we are left with the normalization condition

$$\int_0^\infty dr |u^l(r)|^2 = 1. \tag{9.4.5}$$

We therefore see that $u^l(r)$ is, to all practical purposes (including normalization), a one-dimensional wave function for a particle constrained in the region $0 \leq r < \infty$.

9.4.2 The Small r limit

In the limit of small distances $r \to 0$, the radial equation can be used to find general properties of the radial wave function. We start making the assumption that $u^l(r) \sim r^s$ for small r, where s is some power we want to determine. With this assumption, we have

$$-\frac{\hbar^2}{2M}s(s-1)r^{s-2} + \frac{l(l+1)\hbar^2}{2M}r^{s-2} + V(r)r^s = Er^s. \tag{9.4.6}$$

So far we haven't made any assumption about the central potential itself, but we need to make sure that it has some reasonably good behavior when approaching the origin. Here we make the assumption that the central potential is sufficiently regular, i.e. it can be divergent at the origin, but with a power such that at least

$$\lim_{r\to 0} r^2 V(r) = 0, \tag{9.4.7}$$

notice that this is the case, for example, for the Coulomb potential $V(r) \sim 1/r$. If this is the case, then the dominating term for small values of r is the powers r^{s-2}, thus in order to cancel the divergence in Eq. (9.4.6), we need to have that the terms proportional r^{s-2} cancel out, thus requiring

$$s(s-1) = l(l+1), \tag{9.4.8}$$

which is satisfied for two possible values:

$$s = l + 1 \tag{9.4.9}$$

$$s = -l. \tag{9.4.10}$$

This in turn implies that either $u^l(r) \sim r^{l+1}$ or $u^l(r) \sim \frac{1}{r^l}$, for small distances. However, recalling that $l \geq 0$, we now argue that only the first solution is acceptable. If, for $l \geq 1$, the second possibility is considered, we would then have a divergence at the origin that would violate the normalization condition, which is clearly unacceptable. For $l = 0$, the normalization could still be satisfied, however this would imply that $\Phi^l(r) \sim \frac{1}{r}$, and this comes with other issues. An intuitive argument for this is that the 3-dimensional kinetic energy would then behave as

$$\nabla^2 \frac{1}{r} = -4\pi\delta^3(r), \tag{9.4.11}$$

thus for the energy E to be finite, we would need the central potential to compensate the delta singularity. However, because we are considering here only regular potentials satisfying Eq. (9.4.7), we argue that $u^l(r) \sim \frac{1}{r^l}$ is not an allowed behavior. In conclusion, we must have

$$u^l(r) \sim r^{l+1}, \tag{9.4.12}$$

for small r, and the probability density of finding the particle at the origin is always vanishing, since for $l \geq 0$

$$|u^l(0)|^2 = 0\,. \tag{9.4.13}$$

9.5 Coulomb Potential

We now specialize our discussion to the very important case of a Coulomb potential. We consider here the case in which a single electron (we ignore its spin degrees of freedom) feels the interaction potential due to a nucleus of charge Ze. If we make the assumption that the nucleus is much heavier than the electron, we can neglect the motion of the nucleus and only consider the electron motion in the field of the nucleus. Notice that this assumption is very well verified in practice, since a proton is almost 2000 times heavier than an electron ($m_p/m_e = 1836.15\ldots$). The Hamiltonian then simply reads

$$\hat{H} = \frac{\hat{P}^2}{2m_e} - \frac{Ze^2}{|\hat{r}|}, \tag{9.5.1}$$

where m_e is the electron mass. We therefore see that this is a special case of a central potential, and that the radial equation to be satisfied is

$$\left[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial r^2} + \frac{l(l+1)\hbar^2}{2m_e r^2} - \frac{Ze^2}{r}\right] u^l(r) = E u^l(r). \tag{9.5.2}$$

The potential energy goes to zero at infinity, thus bound states must have negative energy. Focusing then on bound states only, with $E = -|E|$, we can get rid of dimensional factors through the following dimensionless distance:

$$\rho = \sqrt{\frac{8m_e|E|}{\hbar^2}}r. \tag{9.5.3}$$

With this substitution, the radial equation becomes

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\rho^2} - \frac{l(l+1)}{\rho^2}u + \left(\frac{\lambda}{\rho} - \frac{1}{4}\right)u = 0, \tag{9.5.4}$$

where we have also introduced

$$\lambda = \frac{Ze^2}{\hbar}\sqrt{\frac{m_e}{2|E|}}. \tag{9.5.5}$$

The above differential equation is still not easy to solve analytically (it's very easy to solve on a computer though), but we can still deduce the spectrum of eigenvalues just looking at the asymptotic behaviors of the radial wave function.

9.5.1 Behavior at $\rho \to \infty$

In this regime, the above differential equation simplifies to

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\rho^2} - \frac{1}{4}u = 0, \tag{9.5.6}$$

which has solutions

$$u(\rho) = Ae^{-\rho/2} + Be^{\rho/2}. \tag{9.5.7}$$

However, we have previously seen that $u(\rho)$ must be normalizable, thus we must have $B = 0$.

9.5.2 Regular Solutions

Since we know both the large ρ behavior, and the small ρ behavior (from the general discussion on the properties of the radial wave-function), we can attempt to find a solution that satisfies both boundary conditions, of the form

$$u(\rho) = \rho^{l+1} e^{-\rho/2} F(\rho), \tag{9.5.8}$$

where we can expect now $F(\rho)$ to be a smooth, non-singular function. With this further substitution we have

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\rho^2} + \left(\frac{2l+2}{\rho} - 1\right)\frac{\mathrm{d}F}{\mathrm{d}\rho} + \left(\frac{\lambda}{\rho} - \frac{l+1}{\rho}\right) F = 0, \tag{9.5.9}$$

which we can now attempt to solve considering a series expansion for the function F

$$F(\rho) = \sum_{k=0}^{\infty} c_k \rho^k, \tag{9.5.10}$$

with the only constraint that $c_0 \neq 0$, so to guarantee that $u(\rho \to 0) \sim \rho^{l+1}$. With this series expansion, we see that the coefficients c_k must satisfy

$$\sum_{k=2}^{\infty} k(k-1) c_k \rho^{k-2} + \sum_{k=1}^{\infty} (2l+2) k c_k \rho^{k-2} + \sum_{k=0}^{\infty} \left[-k + \lambda - (l+1)\right] c_k \rho^{k-1} = 0. \tag{9.5.11}$$

Making the change of indices $k-1 = k'$ in the first two summations and further renaming $k' \to k$, we get

$$\sum_{k=0}^{\infty} \left[k(k+1) + (2l+2)(k+1)\right] c_{k+1} \rho^{k-1} + \sum_{k=0}^{\infty} \left[-k + \lambda - (l+1)\right] c_k \rho^{k-1} = 0, \tag{9.5.12}$$

which is satisfied if

$$\frac{c_{k+1}}{c_k} = \frac{k + l + 1 - \lambda}{(k+1)(k+2l+2)}. \tag{9.5.13}$$

This equation behaves like

$$\frac{c_{k+1}}{c_k} \sim \frac{1}{k}, \tag{9.5.14}$$

in the limit of large k. However, this is the same asymptotic behavior that you can expect from the function e^ρ, for which $c_k = \frac{1}{k!}$. This behavior is therefore again forbidden, since it would imply that $u(\rho) \sim e^\rho$ and thus the radial wave function would be not normalizable. The only possibility here is therefore that the series expansion terminates at some value of $k_{\max}$ such that $c_k = 0, \forall k > k_{\max}$. The termination condition implies

$$k_{\max} + l + 1 - \lambda = 0, \tag{9.5.15}$$

where $k_{\max} = 0, 1, 2, \ldots$ and recalling the definition of the factor λ in terms of the energy, we finally have that the spectrum of energy eigenvalues is given by

$$E = -\frac{m_e Z^2 e^4}{2\hbar^2 \left(l + 1 + k_{\max}\right)^2}. \tag{9.5.16}$$

This can be further simplified noticing that we can define an integer, called *principal quantum number*, such as

$$n = 1 + l + k_{\max}\,, \tag{9.5.17}$$

thus yielding the quantized eigenvalues

$$E_n = -\frac{m_e Z^2 e^4}{2\hbar^2 n^2}\,, \tag{9.5.18}$$

where now $n = 1, 2, \ldots$. For the Hydrogen atom ($Z = 1$), the numerical constant is given by

$$\frac{m_e e^4}{2\hbar^2} \simeq 13.6\ [\text{eV}]\,. \tag{9.5.19}$$

9.5.3 Examples of Eigenstates

To summarize, the eigenstates are indexed by the principal quantum number as well as by the angular momentum indices. It is also given by

$$\langle r|n,l,m\rangle = \Psi_{nlm}(r) = \Phi_n^l(r) Y_l^m(\theta,\phi) = \frac{u_n^l(r)}{r} Y_l^m(\theta,\phi)\,, \tag{9.5.20}$$

where we have explicitly added the dependency of the radial functions Φ both on l and n, the principal quantum number, since we showed that

$$\begin{aligned} u_n^l(\rho) &= \rho^{l+1} e^{-\rho/2} F(\rho) = \\ &= \rho^{l+1} e^{-\rho/2} \sum_{k=0}^{k_{\max}} c_k \rho^k = \\ &= \rho^{l+1} e^{-\rho/2} \sum_{k=0}^{n-1-l} c_k \rho^k\,. \end{aligned} \tag{9.5.21}$$

The dimensionless radial variable is given by

$$\rho = \sqrt{\frac{8m_e|E|}{\hbar^2}} r = \frac{2}{n}\left(\frac{m_e e^2}{\hbar^2}\right) r = \frac{2r}{n a_0}\,. \tag{9.5.22}$$

where we have introduced a characteristic length scale for the hydrogen atom introduced by Bohr, and known as the Bohr radius:

$$a_0 = \frac{\hbar^2}{m_e e^2} \simeq 0.529\,[\text{Å}] = 52.9\,[\text{pm}]\,. \tag{9.5.23}$$

For example, these expressions allow to easily find the ground state wave function, for which we have $n = 1$ and $l = 0$. The radial part for the ground state reads

$$u_1^0(\rho) = c_0 \rho e^{-\rho/2}, \tag{9.5.24}$$

$$\Phi_1^0(r) = 2\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}, \tag{9.5.25}$$

where the normalization constant c_0 has been explicitly computed imposing

$$\int_0^\infty dr|c_0|^2|u_1^0(r)|^2 = 1\,. \tag{9.5.26}$$

Taking into account also the angular dependence through the spherical harmonic, the full wave function for the ground state is then

$$\Psi_{100}(r) = \Phi_1^0(r)Y_0^0(\theta,\phi), = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}\,. \tag{9.5.27}$$

From this expression we clearly see that Bohr's radius plays the role of a characteristic distance after which the probability of finding the electron is exponentially suppressed. Similarly, other low-energy eigenstates can be explicitly found. We quote here a few of them:

$$\Psi_{200}(r) = \frac{1}{4\sqrt{2\pi a_0^3}} \left(2 - \frac{r}{a_0}\right) e^{-r/2a_0}, \tag{9.5.28}$$

$$\Psi_{210}(r) = \frac{1}{4\sqrt{2\pi a_0^3}} \left(\frac{r}{a_0}\right) e^{-r/2a_0} \cos\theta, \tag{9.5.29}$$

$$\Psi_{21\pm1}(r) = \frac{1}{8\sqrt{\pi a_0^3}} \left(\frac{r}{a_0}\right) e^{-r/2a_0} \sin\theta\,. \tag{9.5.30}$$

9.6 References and Further Reading

The discussion in this Chapter discusses the eigenstates of central potentials, a topic which is found in all quantum mechanics books. For example, Cohen-Tannoudji's book contains, in Chapter 7, a detailed discussion.